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ON THE DERIVATIVES OF A FUNCTION AT A POINT.*

By J. F. Ritt.

1. It was at the suggestion of Professor Kasner that we undertook to find whether to every sequence of finite numbers $a_1, a_2, \cdots, a_n, \cdots$, where $\sqrt[n]{|a_n|/n!}$ need not remain bounded as n increases indefinitely, there corresponds a function of z whose nth derivative for $z = 0$ is a_n; that is, at least in the real domain.

We have since learned that this question had already been settled in the affirmative by Borel† and by Serge Bernstein.‡ Borel's method involves the solution of an infinite system of linear equations. Bernstein employed his normal series for the purpose. These two methods differ entirely from each other and from the method used in this paper. Borel applied his method to the general theory of infinite systems of linear equations. Bernstein considered the case where the derivatives at several points are assigned in advance. We shall arrive in this paper at results which do not seem to flow from the other two solutions. In particular, we shall settle some questions raised several years ago by Van Vleck.§

2. Consider the series

$$\sum u_n = \sum_{n=1}^{\infty} \frac{a_n z^n}{n!} (1 - e^{-(1/b_n z^2)})^{p_n},$$

where p_n, which may vary with n, is any integer not less than unity and where $p_n < b_n > 2^{p_n-1}|a_n|$.

Observe that while each a_n may be any complex number, the above inequality requires each b_n to be real and positive.

To obtain a function with the desired derivatives, it would suffice to take $p_n = 1$ for every value of n, and the convergence discussions which follow would become extremely simple. Still we shall discuss the more general case in full, as it will lead to an interesting result.

We shall consider first the behavior of Σu_n in the domain composed of those points $z = x + yi$ for which $y^2 \leqq m^2 x^2$, where $m^2 < 1$. In this sector, which is limited by the lines $y = mx$ and $y = -mx$, and which contains the axis of real numbers, we have

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* Read before the American Mathematical Society, Feb. 27, 1915, and Dec. 27, 1915.

† Borel, "Sur quelques points de la théorie des fonctions," Ann. de l'Ec. Norm., 1895, p. 38.

‡ D'Adhémar, "Principes de L'Analyse," vol. II, p. 272. I am indebted to Dr. G. A. Pfeiffer for this reference.

§ Van Vleck, "Divergent series and continued fractions," The Boston Colloquium Lectures, p. 84.

$$\cos 2\theta \geqq \frac{1 - m^2}{1 + m^2},$$

where θ is the amplitude of z. Incidentally, the real part of $1/z^2$ is positive.

One sees readily that in this sector the Cauchy function $e^{-(1/z^2)}$ approaches zero uniformly with z, and has zero derivatives of all orders at the origin, if it is given there the value zero.

We shall show now that Σu_n is convergent in the domain under consideration, and that all its derived series are uniformly convergent in every portion of that domain in which $|z|$ is bounded, so that, since

$$u_n = \frac{a_n z^n}{n!}\left[1 - p_n e^{-(1/b_n z^2)} + \frac{p_n(p_n - 1)}{2!} e^{-(2/b_n z^2)} - \cdots + (-1)^{p_n} e^{-(p_n/b_n z^2)}\right],$$

Σu_n has the desired derivatives for $z = 0$.*

We use, for this purpose, the inequalities

$$|1 - e^{-(1/B)}| < 2$$

when the real part of B is positive, or when $|B| > 1$, and

$$|A(1 - e^{-(1/B)})| < 2$$

when the real part of B is positive and $|B| > |A|$, or when $1 < |B| > |A|$.

The first of these is obvious when the real part of B is positive, for then $|e^{-(1/B)}| < 1$. Hence the second is true when the real part of B is positive and when $|A| < 1$. When $|B| > 1$,

$$(1 - e^{-(1/B)}) = \left|\frac{1}{B} - \frac{1}{2!B^2} + \cdots\right| < \frac{e - 1}{B} < \frac{2}{B},$$

and this fact accounts for the remaining cases.

Now

$$|u_n| = \left|\frac{z^{n-2}}{n!} 2^{p_n-1} a_n z^2 (1 - e^{-(1/b_n z^2)}) \frac{(1 - e^{-(1/b_n z^2)})^{p_n-1}}{2^{p_n-1}}\right|,$$

and since the real part of $b_n z^2$ is positive and $b_n z^2 > 2^{p_n-1} a_n z^2$, we have by the inequalities above,

$$|u_n| < 2\left|\frac{z^{n-2}}{n!}\right|,$$

from which it is evident that Σu_n is uniformly convergent in any bounded portion of our domain.

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* The sufficiency of these conditions is generally proved only for real functions of a real variable. It is not difficult to show that they suffice also in the present case.

Take now the first derived series

$$\sum u_n' = \sum_{n=1}^{\infty}\left[\frac{a_n z^{n-1}}{(n-1)!}\left(1 - e^{-(1/b_n z^2)}\right)^{p_n} - \frac{2p_n a_n z^{n-3}}{b_n n!}\left(1 - e^{-(1/b_n z^2)}\right)^{p_n-1} e^{-(1/b_n z^2)}\right].$$

Consider the four series

$$\sum \alpha_n = \sum_{n=1}^{\infty}\frac{a_n z^{n-1}}{(n-1)!}\left(1 - e^{-(1/b_n z^2)}\right)^{p_n}, \qquad \sum \beta_n = \sum_{n=1}^{\infty}\frac{p_n a_n}{b_n}\frac{z^{n-3}}{n!}\left(1 - e^{-(1/b_n z^2)}\right)^{p_n},$$

$$\Sigma \gamma_n = \sum_{n=1}^{\infty}\frac{r_n a_n}{b_n}\frac{z^{n-3}}{n!}\left(1 - e^{-(1/b_n z^2)}\right)^{p_n-1}, \qquad \Sigma \delta_n = \sum_{n=1}^{\infty}\frac{s_n a_n}{b_n}\frac{z^{n-3}}{n!},$$

where $r_n = p_n$ or 0 according as $p_n > 1$ or $p_n = 1$, and $s_n = 0$ or 1 according as $p_n > 1$ or $p_n = 1$.

The series $\Sigma \alpha_n$, $\Sigma \beta_n$ and $\Sigma \gamma_n$ do not differ materially in form from Σu_n. This is quite evident in the case of $\Sigma \alpha_n$. It becomes apparent also for $\Sigma \beta_n$ and $\Sigma \gamma_n$ if we ignore the first three terms of those series, and consider that, since $b_n > p_n$, $|p_n a_n / n(n-1)(n-2) b_n| < |a_n| < b_n / 2^{p_n - 1}$. Hence any property of Σu_n which arises from the fact that $p_n < b_n > 2^{p_n-1}|a_n|$ is enjoyed also by $\Sigma \alpha_n$, $\Sigma \beta_n$ and $\Sigma \gamma_n$. The series $\Sigma \delta_n$, if we drop its first two terms, becomes an entire function.

Hence, we may write

$$\Sigma u_n' = \Sigma \alpha_n + 2\Sigma \beta_n - 2\Sigma \gamma_n - 2\Sigma \delta_n,$$

so that $\Sigma u_n'$ converges uniformly wherever Σu_n does, and represents the derivative of Σu_n. The series $\Sigma \delta_n$ can be differentiated formally any number of times, and the series $\Sigma \alpha_n$, $\Sigma \beta_n$ and $\Sigma \gamma_n$, like Σu_n, at least once. Hence the second derivative of Σu_n exists and can be obtained formally. Thus, if Σu_n can be differentiated formally n times, the next formal differentiation will also be valid. As remarked above, it is evident that the nth derivative, for $z = 0$, is a_n.* It should be emphasized that when we speak of derivatives at the origin, we understand that z is confined to the sector described above.†

Similarly, the function

$$\sum_{n=1}^{\infty}\frac{a_n z^n}{n!}\left(1 - e^{-(1/b_n y)}\right),$$

where $1 < b_n > |a_n|$ and $y = z$ or $-z$, according as the real part of z is positive or is negative, will have the desired derivatives for $z = 0$. Here our domain is the entire z-plane, excepting those points for which $\cos^2 \theta < \epsilon$, θ being the amplitude of z and ϵ an arbitrarily small positive number. Thus while, when lim. sup. $\sqrt[n]{|a_n|/n!} = \infty$, there exists no function which has the desired derivatives relative to a domain of which the origin is an interior point, the above example permits us to come as close as we desire to a complete envelopment of the origin. By using the function $e^{-(1/\sqrt{z})}$ one could hope to cover one of the two narrow gaps still left in our domain, but we shall not go into the convergence investigations.

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* Evidently the problem of finding the most general function with the desired derivatives is identical with that of finding the most general function with zero derivatives.

† In general the sides of the sector cannot be rotated beyond the lines $y = x$ and $y = -x$, without destroying the regularity of the behavior of Σu_n. For, if the derivatives should happen to exist relative to a sector of opening greater than $\pi/2$, a slight change of one of the b's would destroy the regularity outside of the lines $y = x$ and $y = -x$. Cf. Van Vleck, loc. cit.

3. We wish to show that the function Σu_n of § 1 can be developed in a Laurent series, convergent everywhere except at the origin. For this purpose we shall show that Σu_n is uniformly convergent within the ring formed by two circles, with centers at the origin, and with any radii $r > 0$ and $R > r$ respectively.

If $b_n r^2 \leqq 1$, we have when $|z| > r$, since $|a_n| < b_n > p_n$,

$$|u_n| < \frac{(1 + e^{1/r^2})^{1/r^2}}{r^2} \left| \frac{z^n}{n!} \right| .$$

If $b_n r^2 > 1$, we have, by the inequalities of § 1,

$$|u_n| < 2 \left| \frac{z^{n-2}}{n!} \right| < \frac{2}{r^2} \left| \frac{z^n}{n!} \right| .$$

From the above upper limits for $|u_n|$ the uniform convergence of Σu_n within the ring in question is apparent. Hence Σu_n admits of development in a Laurent series, convergent everywhere except at the origin, which can be gotten by adding up the developments of the terms u_n. Let

$$\sum u_n = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{n=1}^{\infty} \beta_n z^{-n}.$$

If lim. sup. $\sqrt[n]{|a_n|/n!} = \infty$, the principal parts of the derivatives are evidently contributed by the negative power series $\Sigma \beta_n z^{-n}$. One naturally inquires as to whether one can find a series of *negative powers alone* which has any desired derivatives at the origin. Such a series we shall in fact obtain by taking each p_n sufficiently large, since the exponent of the highest power of z, in u_n, is $n - 2p_n$.

Having obtained such a series of negative powers, we can, by adding terms of the form $Kz^{-n}e^{-1/z^2}$, reduce any finite number of coefficients to any values desired, without affecting the derivatives at the origin. Thus, being given any sequence of numbers, $a_1, a_2, \cdots, a_n, \cdots$, and any function $\sum_{n=1}^{n=m} \beta_n z^{-n}$, we can, in an infinite number of ways, determine a series $\sum_{n=m+1}^{\infty} \beta_n z^{-n}$, such that $\sum_{n=1}^{\infty} \beta_n z^{-n}$ has a_n as its nth derivative at $z = 0$, provided z remains in the domain described in connection with the function e^{-1/z^2}.

The delicate adjustment of coefficients which, when z remains in a limited domain, gives the series $\Sigma \beta_n z^{-n}$ a finite value and definite derivatives of all orders for $z = 0$, at which point each term $\beta_n z^{-n}$ becomes infinite, suggests interesting investigations. However, all our attempts to discover the nature of this adjustment have thus far been without significant result.

4. We shall extend the results of § 1 to the case of n variables, each variable remaining in a sector similar to that described in connection with the function e^{-1/z^2}. That is, assuming all differentiations commutative, we shall show the existence of a function with any desired partial derivatives for $x_1 = x_2 = \cdots = x_n = 0$. For simplicity of notation we limit ourselves to two variables, but the generality of the principle employed will be apparent.

Consider the series

$$\sum u = \sum \frac{a_{p,q} x_1^p x_2^q}{p!q!} \left(1 - e^{-\frac{1}{b_{p,q} x_i^2}}\right),$$

where i is to be either 1 or 2, and where $1 < b_{p,q} > N \mid a_{p,q} \mid$, N being the number of terms in a homogeneous polynomial of degree $p + q$. The significance of $a_{p,q}$ is evident. We wish to choose i in each term in such a way that there may be only a finite number of terms in which a factor $1 - e^{-1/b_{p,q} x_1^2}$ occurs, and in which the exponent of x_1 is less than any positive integer M; similarly for x_2. The purpose of this is that only a finite number of negative powers may be introduced at each differentiation. It will suffice, for instance, to take $i = 1$ if $p \geqq q$ and $i = 2$ if $p < q$.

Writing

$$\sum | u_n | = \sum_{n=1}^{\infty} \sum_{p=0}^{p=n} \frac{\left| a_{n-p,p} x_1^{n-p} x_2^p \left(1 - e^{-\frac{1}{b_{n-p,p} x_i^2}}\right) \right|}{(n-p)!p!},$$

one recognizes readily that Σu_n is absolutely and uniformly convergent, within any finite limits, provided x_1 and x_2 remain in the domains described above. The remainder of the discussion will be similar to that in the case of one variable if one separates Σu into two series, the first formed of those terms which contain $1 - e^{-1/b_{p,q} x_1^2}$ as a factor, the second formed of the remaining terms. Corresponding to the power series $\Sigma \delta_n$ there will appear two power series in two variables whose majorants are everywhere convergent. All derivatives will be continuous with respect to both variables. Hence, as far as the real domain is concerned, the differentiations will be commutative.* But since the derivatives in the real domain determine the derivatives in the entire domain under consideration, the differentiations in the entire domain are commutative.

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* Of course, if the derivatives are complex, we must consider the real parts and imaginary parts separately.